PROPOSITION 4. *If R is semiprime and $x \in R$, then x has* LS *and* RS *simultaneously; if some x in R has both* LB *and* RB, *then R is unital.*

Thus for a semiprime R there are five possible combinations: $\varnothing$, {LB}, {RB}, {LS, RS}, Δ.

Full details will appear in [3].

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Products of Skew-Symmetric Matrices

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This note summarizes some recent results obtained jointly by R. Gow and the author. Detailed proofs will appear elsewhere.

Throughout this note, F denotes an algebraically closed field of characteristic not equal to 2, and $A \in M_n(F)$ is a nonsingular $n \times n$ matrix.

We consider the following questions:

(1) Can A be expressed as a product of a finite number k of skew-symmetric matrices in $M_n(F)$?

(2) If so, what is the smallest number k for which such a representation is possible?

Note that since A is nonsingular, a necessary condition is that n is even. Also, the product of two skew-symmetric 2×2 matrices is scalar, so a 2×2 matrix A satisfies (1) if and only if it is either skew-symmetric or scalar.

We now state our main results.

THEOREM 1 [2]. *A is a product of two skew-symmetric matrices if and only if A is similar to a matrix of the form*

$$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

THEOREM 2. *Suppose* $n \equiv 0 \bmod 4$. *Then* A *can be expressed as a product of* five *skew-symmetric matrices. There is a nonsingular matrix* A *which cannot be expressed as a product of fewer than five skew-symmetric matrices.*

THEOREM 3. *Suppose* $n > 2$ *and* $n \equiv 2 \bmod 4$. *Then* A *can be expressed as a product of* seven *skew-symmetric matrices.*

REMARK. We do not know whether "seven" can be replaced by "five" in Theorem 3.

We now sketch the proof of Theorem 2. By a result of Gow [1], $A = SJ$, where $S \in M_n(F)$ is symmetric and J is an involution, i.e., $J^2 = I$. A careful analysis of the proof shows that J may be chosen to have $n/2$ eigenvalues equal to $+1$ and $n/2$ eigenvalues equal to -1. Since $n \equiv 0 \bmod 4$, it thus follows that J is similar to the matrix of the form

$$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \qquad \text{where} \quad B = \begin{bmatrix} I_{n/4} & 0 \\ 0 & -I_{n/4} \end{bmatrix},$$

and hence, by Theorem 1, J is a product of two skew-symmetric matrices. Next, since F is algebraically closed, there is a nonsingular matrix P in $M_n(F)$ such that

$$P'SP = I.$$

Let K be a nonsingular skew-symmetric matrix in $M_{n/2}(F)$. Then

$$\begin{bmatrix} K^{-1} & 0 \\ 0 & K^{-1} \end{bmatrix}$$

is a product of two skew-symmetric matrices by Theorem 1, and hence $I = S_1S_2S_3$ is a product of three skew-symmetric matrices. Hence

$$S = \left(P'^{-1}S_1P^{-1}\right)\left(PS_2P'\right)\left(P'^{-1}S_3P^{-1}\right)$$

is a product of three skew-symmetric matrices, and thus A is a product of five skew-symmetric matrices.

We next show that there is a nonsingular matrix A in $M_n(F)$ (n even) which cannot be expressed as a product of four skew-symmetric matrices. Suppose $A = P_1P_2P_3P_4$ where P_1, P_2, P_3, P_4 are skew-symmetric. By Theorem 1, $A = XY$ where X, Y have each of their elementary divisors occurring with even multiplicity. Taking A to be a diagonal matrix $\mathrm{diag}(1,1,\dots,1,a)$ where $a \neq 0, 1$, we use the Sá-Thompson interlacing theorem for invariant factors [3, 4] to get a contradiction. This shows that there is an A which is not a product of four skew-symmetric matrices. Hence there is a nonsingular A_0 which cannot be written as a product of three skew-symmetrics (for example, take $A_0 = AK^{-1}$ where K is a skew-symmetric and A cannot be written as a product of four skew-symmetric matrices). It then follows that there is a nonsingular A_1 which cannot be expressed as a product of fewer than five skew-symmetric matrices.

The proof of Theorem 3 is similar in style to that of Theorem 2, but more difficult because J does not satisfy Theorem 1. The key step is to show that if $K \in M_n(F)$, $n > 2$, $n \equiv 2 \bmod 4$, is nonsingular and skew-symmetric, then K can be expressed as a product of four skew-symmetrics. This is done by examination of the Jordan form of K; an inductive argument reduces the problem to two cases:

(1) K similar to $J_{n/2}(\lambda) \oplus J_{n/2}(-\lambda)$,
(2) K similar to

$$J_{(n-2)/2}(\lambda) \oplus J_{(n-2)/2}(-\lambda) \oplus \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

[where $J_r(\lambda)$ denotes the $r \times r$ Jordan block with eigenvalue λ], and these are dealt with by direct computation.

We have extended some of the results to more general fields F (assuming $\det A$ is a square in F)—the details will appear elsewhere.

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